Midterm

Introduction to Computational Logic | Fall 2024

(1) Find natural deduction proofs for the following sequents:

(a) $p \implies q \vdash \neg p \vee q$. (10%)

(b) $\vdash ((p \implies q) \implies p) \implies p$. (10%)

(2) A *cube* is a conjunction of literals. A formula is in *disjunctive normal form (DNF)* if it is a disjunction of cubes.

(a) For any propositional logic formula ϕ, find a propositional logic formula ψ in DNF such that $\phi \equiv \psi$. (10%)

(b) Show a cube is not satisfiable if and only if there is a propositional atom p such that both p and $\neg p$ appear in the cube. (10%)

(c) Given a formula ϕ in DNF, please find a polynomial-time algorithm to determine if ϕ is satisfiable. (10%)

(3) Answer the following questions:

(a) Find a natural deduction proof for $\exists x \forall y \neg \phi \vdash \neg \forall x \exists y \phi$. (10%)

(b) Show there is no natural deduction proof for $\forall y\ \exists x\ P(x, y) \vdash \exists x\ \forall y\ P(x, y)$. (10%)

(4) Answer the following questions and justify your answers:

(a) Is there a predicate logic sentence ϕ such that (10%)

for every model $\mathcal{M}$, $\mathcal{M} \models \phi$ iff the size of the universe of $\mathcal{M}$ is a multiple of 3 less then 2024?

(b) Is there a predicate logic sentence ψ such that (10%)

for every model $\mathcal{M}$, $\mathcal{M} \models \psi$ iff the size of the universe of $\mathcal{M}$ is a multiple of 3?

finite

infinite

Please turn the page for one more question!

$\frac{n(n-1)}{2}$

(5) Let 0 be a constant symbol, S a unary function symbol, and $<$ a binary predicate symbol. Define

$$\Gamma = \left\{ \begin{array}{c} \forall x \neg S(x) = 0, \\ \forall x \forall y (S(x) = S(y) \implies x = y), \\ \forall y (\neg y = 0 \implies \exists x y = S(x)), \\ \forall x \neg S(x) = x, \forall x \neg S^2(x) = x, \ldots, \forall x \neg S^n(x) = x, \ldots \end{array} \right\} \quad S^i(x) \neq x \ \forall i$$

(a) Find a model $\mathcal{M}$ for Γ. (10%)

(b) Write $x < y$ for $< (x, y)$ and $\phi \Leftrightarrow \psi$ for $(\phi \implies \psi) \wedge (\psi \implies \phi)$. Let

$$\Delta = \left\{ \begin{array}{c} \forall x \forall y (x < S(y) \Leftrightarrow x = y \vee x < y), \\ \forall x \neg x < 0, \\ \forall x \forall y (x < y \vee x = y \vee y < x), \\ \forall x \forall y (x < y \implies \neg (y < x)), \\ \forall x \forall y \forall z (x < y \implies (y < z \implies x < z)) \end{array} \right\} \quad \forall x,\ x \geq 0$$

Find a model $\mathcal{N}$ for $\Gamma \cup \Delta$. (10%)

(c) Find a model $\mathcal{K}$ such that there is a c in the universe of $\mathcal{K}$ such that for any $n \geq 0$

$$\overbrace{S^{\mathcal{K}}(S^{\mathcal{K}}(\cdots S^{\mathcal{K}}(}^{n} \cdots (0^{\mathcal{K}}) \cdots) \neq c.$$

0, 1, 3, 5 . . . .

(10%)