Programming Languages: Imperative Program Construction Midterm

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$p \equiv \neg p \equiv \cancel{(p \vee \neg p)}$

Notes regarding proofs.

- Questions 1 and 2 are about logic. Thus for each step you need to write down a "reason", that is, what axiom/theorems you use.
  - However, properties such as symmetry associativity, zero, identity of logical operators (e.g. (3.24), (3.25), (3.36), (3.37), (3.29), (3.30)) are too ubiquitous and can be used without explicit mentioning.
  - You can skip some steps, or combine several steps into one, as long as you think you can convince me the steps are correct.
- For questions 4(b-d) and 5 you need to do some calculation, but you do not need to write down the "reasons" for each steps.
- Arithmetic properties (e.g. those regarding $+, -, \times, \leqslant, \geqslant$, etc) are assumed to be known and can be used without mentioning.

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1. (10 points) Prove (3.63) $p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$, using properties that appear before (3.63).

2. (10 points) Prove that $\neg\, p \Rightarrow (p \Rightarrow q)$. **Hint**: there are many possible proofs. In some proofs you might try to reduce the entire expression to *True*.

3. (a) (5 points) Let N be an *Int* (integer) such that $N \geqslant 0$, and A an array of *Int* containing N elements, indexed by $A\,[0], A\,[1] \ldots A\,[N-1]$ (if these elements exist).

   For i, j such that $0 \leqslant i \leqslant j \leqslant N$, we denote by $A[i..j)$ a consecutive segment of an array that includes $A\,[i]$ but does not include $A\,[j]$. For example, if $N \geqslant 10$, by $A[2..10)$ we denote the segment $A\,[2], A\,[3] .. A\,[9]$. If $i = j$, the segment is empty.

   Assuming that $0 \leqslant i \leqslant j \leqslant N$, write down an expression stating that "s is the sum of $A[i..j)$." $A[i] \sim A[j-1]$

   (b) (10 points) A consecutive segment of an array of *Int* is called "steep" (陡 in Chinese) if each of its elements is larger than the sum of all elements to its lefthand side. For example, in the array below,

   6, 3, 4, 8, 10, 19, 38, 2, 7,

   the segment 3, 4, 8 is steep (since $0 < 3$, $3 < 4$ and $3 + 4 < 8$), the segments 8, 10, 19, 38 and 2, 7 are also steep (since $8 < 10$, $8 + 10 < 19$, $8 + 10 + 19 < 38$, etc). An empty segment is steep. A singleton segment containing one negative element, for example, -1, is *not* steep, since -1 is not larger than the sum of all elements to its lefthand side, which is 0.

   Assuming that $0 \leqslant i \leqslant j \leqslant N$, write down an expression stating that "b is true if and only if $A[i..j)$ is steep."

   (c) (10 points) Write down an expression stating that "r is the length of the longest steep segment of the array A."

4. Consider the following program

$$\begin{array}{l} \textbf{if } x > 3 \rightarrow skip \\ \mid x < 0 \rightarrow x := -2 \times x \\ \textbf{fi} \end{array}$$

Denote this program by *PROG*.

(a) (10 points) Write down *wp PROG q*.

(b) (10 points) What is the weakest precondition for *PROG* to terminate?

(c) (10 points) What is *wp PROG* $(x > 4)$? (You may use your knowledge about arithmetics to simplify the ranges.)

(d) (10 points) What is *wp PROG False*? $\rightarrow 0 \leq \cancel{x \leq 3}$

5. (15 points) Prove the following Hoare triple:

$$\begin{array}{l} \{3 \leqslant x \vee (-1 \leqslant x < 0)\} \\ \textbf{if } 0 < x \rightarrow x := x - 1 \\ \mid x < 0 \rightarrow x := x + 3 \\ \textbf{fi} \\ \{1 \leqslant x\} \ . \end{array}$$

$x > 3 \Rightarrow x > 4 \quad x \leq 3 \vee x > 4$

$x < 0 \Rightarrow x < -2 \quad x \geq 0 \vee x < -2$

$x > 3 \vee x < 0$

$1 \leq x - 1$

$2 \leq x$

$1 \leq x + 3$

$-2 \leq x$

$\sim p \sim$

$\Rightarrow \sim q \sim$

$p \Rightarrow q$

$-2x > 4$

$x < -2$